

Compactified Jacobians and q, t -Catalan numbers, II

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Abstract

We continue the combinatorial study of the homology of compactified Jacobians of plane curve singularities with one Puiseux pair (m, n) and its relation to the generalized q, t -Catalan numbers. We describe two generalizations of the bijective constructions of J. Haglund and N. Loehr and use them to prove a simple formula for the Poincaré polynomials for the homology for $m = kn \pm 1$. Using a construction of B. Fantechi, L. Göttsche and D. van Straten, we give a bijective proof of the (q, t) -symmetry for $n \leq 3$. We also give a geometric interpretation of a relation to the theory of (m, n) -cores discovered by J. Anderson.

1 Introduction

A. Beauville proved in [5] that the compactified Jacobian of a complete rational curve is homeomorphic to the direct product of Jacobi factors which depend only on the local analytic structure of the singularities of the curve. The Jacobi factor of a singularity can be defined as a subvariety in a Grassmannian in following manner.

Let $x \in C \subset \mathbb{C}^2$ be a unibranched plane curve singularity, t be a normalizing parameter on C at x , and $R \subset \mathbb{C}[[t]]$ be the complete local ring at x . Let $\delta = \dim(\mathbb{C}[[t]]/R)$. Since $x \in C$ is a plane curve singularity, it follows that $t^{2\delta}\mathbb{C}[[t]] \subset R$. Let $V = \mathbb{C}[[t]]/t^{2\delta}\mathbb{C}[[t]]$.

Definition 1.1. The *Jacobi factor* \overline{JC}_x is the space of R -submodules $M \subset \mathbb{C}[[t]]$, such that $M \supset t^{2\delta}\mathbb{C}[[t]]$ and $\dim(\mathbb{C}[[t]]/M) = \delta$.

In other words, \overline{JC}_x is isomorphic to the subvariety of the Grassmannian $Gr(\delta, V)$, consisting of subspaces invariant under R -action.

To study the topology of the compactified Jacobian of a singular rational curve it is sufficient to study the topology of a single Jacobi factor. This problem is quite complicated for a general singularity. In [27] J. Piontowski showed that in some cases the Jacobi factor admits an algebraic cell decomposition. Following his work, we gave a combinatorial description of the cell decomposition in the case of a plane curve singularity with one Puiseux pair. The following theorem is the main result of [12].

Theorem 1.1. ([12]) *Suppose that a plane curve singularity has one Puiseux pair (m, n) . Then its Jacobi factor admits an affine cell decomposition. The cells are parametrized by Young diagrams D contained in $m \times n$ rectangle below the diagonal. The dimension for the cell C_D in the Jacobi factor can be computed in terms of D as follows:*

$$\dim C_D = \frac{(m-1)(n-1)}{2} - h_+^{\frac{m}{n}}(D),$$

where

$$h_+^{\frac{m}{n}}(D) = \# \left\{ c \in D \mid \frac{a(c)}{l(c)+1} \leq \frac{n}{m} < \frac{a(c)+1}{l(c)} \right\},$$

and $a(c)$ and $l(c)$ denote the arm- and leg-length for a box $c \in D$.

For the reader's convenience, from now on we will write h_+ instead of $h_+^{\frac{m}{n}}$.

Motivated by constructions from [16], we introduce the bivariate function

$$c_{m,n}(q, t) = \sum_D q^{\delta - |D|} t^{h_+(D)},$$

where $\delta = \frac{(m-1)(n-1)}{2}$ is the classical δ -invariant of the singularity, and the sum is taken over all diagrams D in $m \times n$ rectangle below the diagonal.

Remark 1.1. *The geometric meaning of $h_+(D)$ is explained in the previous theorem, but the geometric meaning of $|D|$ is far less obvious. It is expected to be related to the perverse filtration on the homology of the Jacobi factor defined via a versal deformation of the singularity (see [23], [22], [25] for more details).*

It has been proved in [10] that $c_{m,n}(q, t)$ coincides with the q, t -Catalan numbers of A. Garsia and M. Haiman ([9]) for $m = n + 1$, and with the

k -analogue of them for $m = kn + 1$. The latter statement holds modulo some conjectures of N. Loehr.

Motivated by these coincidences, we formulate two general conjectures about $c_{m,n}(q, t)$. The second conjecture is a weak version of the first one, both are supported by a vast amount of experimental data available by request to the authors.

Conjecture 1.1 (Symmetry conjecture). *The function $c_{m,n}(q, t)$ satisfies the functional equation*

$$c_{m,n}(q, t) = c_{m,n}(t, q).$$

In [10] this conjecture was proved for $m = n + 1$, where it was deduced from some notrivial identities for q, t -Catalan numbers. For $m = kn + 1$ a similar statement was conjectured in [20]. No bijective proof in any of these cases is known yet.

Theorem 1.2. *The symmetry conjecture holds for $n \leq 3$.*

In the proof we construct an explicit bijection exchanging the area and h_+ statistics. This bijection is obtained as the composition of three bijections, and as an intermediate step we use a monomial basis in a certain algebra constructed by B. Fantechi, L. Göttsche and D. van Straten.

Conjecture 1.2 (Weak symmetry conjecture). *The function $c_{m,n}(q, t)$ satisfies the functional equation*

$$c_{m,n}(q, 1) = c_{m,n}(1, q).$$

The weak symmetry property was proved by an explicit bijective construction for $m = n + 1$ by J. Haglund ([16]) and for $m = kn + 1$ by N. Loehr ([20]). Following their ideas and the constructions from [12], we construct two different maps G_m and G_n from the set of Young diagrams below the diagonal to itself.

Theorem 1.3. *The maps G_m and G_n satisfy the following properties:*

1. $|G_m(D)| = |G_n(D)| = \delta - h_+(D)$.
2. If G_m is bijective then G_n is bijective.
3. If $m = n + 1$, then $G_n = G_{n+1}$. We show an example where G_n and G_m are essentially different.

4. If $m = kn + 1$, then G_m coincides with the Haglund–Loehr map, and hence it is bijective.
5. If $m = kn - 1$, then G_m is bijective too.

Using the property (1), one can show that the bijectivity of G_m (or, equivalently, G_n) is sufficient to prove the weak symmetry conjecture.

Corollary 1.1. *The weak symmetry conjecture holds for $m = kn \pm 1$.*

It was pointed out in [12] that the weak symmetry conjecture implies a remarkably simple formula for the Poincaré polynomial of the Jacobi factor:

$$\begin{aligned} P_{\overline{JC}_x}(t) &= \sum_D t^{2 \dim C_D} = \sum_D t^{2(\delta - h_+(D))} = \\ &= t^{2\delta} c_{m,n}(1, t^{-2}) \stackrel{WS}{=} t^{2\delta} c_{m,n}(t^{-2}, 1) = \sum_D t^{2|D|}. \end{aligned}$$

Corollary 1.2. *If $m = kn \pm 1$ then the Poincaré polynomial of the Jacobi factor has the form*

$$P_{\overline{JC}_x}(t) = \sum_D t^{2|D|},$$

Where the summation is done over all Young diagrams in $m \times n$ rectangle below the diagonal.

Following the ideas of J. Anderson ([3]), we prove the following result. There is a canonical embedding of the Jacobi factor into the Grassmannian $Gr(\delta, V)$. The vector space $V = \mathbb{C}[[t]]/t^{2\delta}\mathbb{C}[[t]]$ comes with a natural filtration

$$V = V_{2\delta} \supset V_{2\delta-1} \supset \dots \supset V_0 = 0, \quad V_i := t^{2\delta-i}V.$$

Therefore $Gr(\delta, V)$ has a cell decomposition by Schubert cells enumerated by Young diagrams contained in a $\delta \times \delta$ square. The cell decomposition of \overline{JC}_x is given by intersections with the Schubert cells ([27],[14]).

Theorem 1.4. *A cell in $Gr(\delta, V)$ has non-empty intersection with \overline{JC}_x if and only if the corresponding Young diagram is a simultaneous (m, n) -core, i.e. it has no hooks of length m or n .*

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2 Homology of the compactified Jacobian

2.1 Cell decomposition

Let $\Gamma = \Gamma_{m,n} \subset \mathbb{Z}_{\geq 0}$ be the semigroup generated by m and n . A subset $\Delta \subset \mathbb{Z}$ is called a semi-module over Γ , if $\Delta + \Gamma \subset \Delta$. It is zero-normalized, if $\min(\Delta) = 0$.

A number a is called a n -generator of Δ , if $a \in \Delta$ and $a - n \notin \Delta$.

Theorem 2.1. ([27]) *The Jacobi factor for a plane curve singularity with one Pusieux pair (m, n) admits an affine cell decomposition. The cells C_Δ are enumerated by the 0-normalized Γ -semimodules Δ , and their dimensions can be computed by the formula*

$$\dim C_\Delta = \sum_{j=1}^n g(a_j) := \sum_{j=1}^n \#([a_j, a_j + m) \setminus \Delta), \quad (1)$$

where a_j are the n -generators of Δ .

It has been remarked in [12] that this definition has a natural combinatorial interpretation. Let us label the (x, y) box of the positive quadrant by the number $mn - m(1 + x) - n(1 + y)$. The corner box $(0, 0)$ is labelled by $2\delta - 1 = mn - m - n = \max(\mathbb{N} \setminus \Gamma)$, and the numbers decrease by m in east direction and by n in north direction. One can check that every number from $\mathbb{N} \setminus \Gamma$ appears exactly once in the $m \times n$ rectangle below the diagonal.

Given a Γ -semimodule Δ , let us mark the boxes labelled by numbers from $\Delta \setminus \Gamma$ and denote the resulting set of boxes by $D(\Delta)$.

Theorem 2.2. ([12]) *For a Γ -semimodule Δ the set $D(\Delta)$ is a Young diagram. The correspondence D between semimodules and Young diagrams*

below the diagonal is bijective. The dimension of a cell C_Δ in the Jacobi factor can be computed in terms of $D(\Delta)$ as follows:

$$\dim C_\Delta = \frac{(m-1)(n-1)}{2} - h_+(D(\Delta)). \quad (2)$$

Remark 2.1. The formula (2) is manifestly symmetric in m and n while (1) is not.

2.2 Duality

Let us recall the useful symmetry property for the plane curve semigroup Γ .

Lemma 2.1. (e.g. [19],[6]) Let Γ be the semigroup of a unibranched plane curve singularity and let δ be its δ -invariant. Then

$$a \in \Gamma \iff 2\delta - 1 - a \notin \Gamma.$$

For every Γ -semimodule Δ we can consider the *dual semimodule*

$$\Delta^* := \{\phi \mid \phi + \Delta \subset \Gamma\}.$$

Lemma 2.2. The dual semimodule Δ^* can be characterised by the equation

$$\Delta^* = (2\delta - 1) - (\mathbb{Z} \setminus \Delta).$$

Proof. Note that $(2\delta - 1) - (\mathbb{Z} \setminus \Delta)$ is a Γ -semimodule. Indeed, if $x \in \mathbb{Z} \setminus \Delta$ then $x - m, x - n \in \mathbb{Z} \setminus \Delta$.

It follows from Lemma 2.1 that

$$\phi + \Delta \subset \Gamma \Leftrightarrow (2\delta - 1) \notin \phi + \Delta.$$

Indeed, if $(2\delta - 1) \in \phi + \Delta$ then $\phi + \Delta$ is not a subset in Γ . Conversely, if $\exists x \in (\phi + \Delta) \setminus \Gamma$, then by Lemma 2.1 $(2\delta - 1 - x) \in \Gamma$, so $(2\delta - 1) \in x + \Gamma \subset \phi + \Delta$. Therefore

$$\begin{aligned} \phi + \Delta \subset \Gamma &\Leftrightarrow (2\delta - 1) \notin \phi + \Delta \Leftrightarrow \\ &\Leftrightarrow (2\delta - 1 - \phi) \notin \Delta \Leftrightarrow \phi \in (2\delta - 1) - (\mathbb{Z} \setminus \Delta). \end{aligned}$$

□

Remark 2.2. Lemmas 2.2 and 2.1 are combinatorial analogues of the Gorenstein property for the plane curve singularities (e.g. [19],[6],[28]).

Example 2.1. Let us list the combinatorial types of Γ -semimodules Δ and Δ^* for $(m, n) = (3, 4)$. We normalize Δ^* so that it starts from 0 and denote the normalized semimodule by $\hat{\Delta}$.

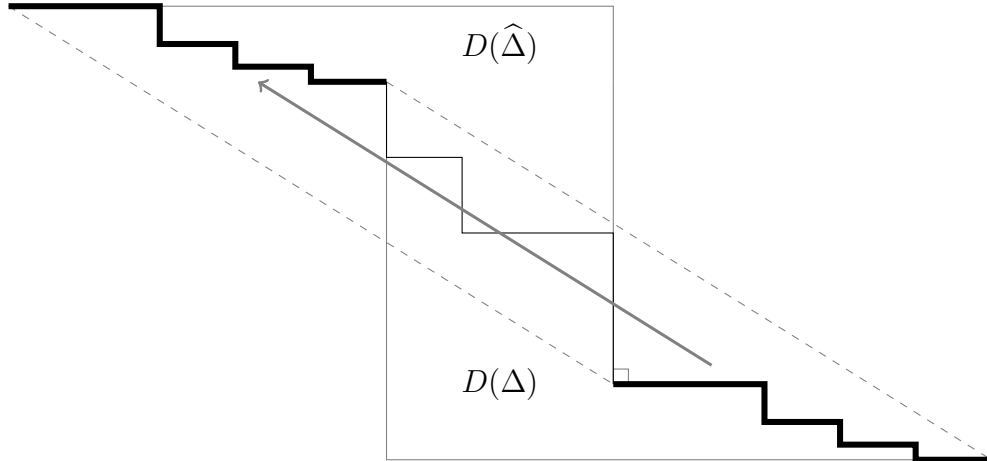
Δ	Δ^*	$\hat{\Delta}$
0, 3, 4, 6, ...	0, 3, 4, 6, ...	0, 3, 4, 6, ...
0, 3, 4, 5, 6, ...	3, 4, 6, ...	0, 1, 3, 4, 5, 6, ...
0, 2, 3, 4, 5, 6, ...	4, 6, ...	0, 2, 3, 4, 5, 6, ...
0, 1, 3, 4, 5, 6, ...	3, 6, ...	0, 3, 4, 5, 6, ...
0, 1, 2, 3, 4, 5, 6, ...	6, ...	0, 1, 2, 3, 4, 6, ...

Let us explain how to get the diagram $D(\hat{\Delta})$ given the diagram $D(\Delta)$. Following Lemma 2.2, we have

$$\hat{\Delta} = \max(\mathbb{Z} \setminus \Delta) - (\mathbb{Z} \setminus \Delta).$$

Consider the Dyck path bounding the diagram $D(\Delta)$ from above. The furthestmost from the diagonal south-west corner of the path splits it in two parts. Note that the box in this corner is labelled by $\max(\mathbb{Z} \setminus \Delta)$. Let us change the order of the parts. For instance, let us shift the south-eastern part by n to the north and by m to the west. We get another Dyck path. $D(\hat{\Delta})$ is the diagram bounded by the new Dyck path from below. Note that because of our choice of the splitting corner, $D(\hat{\Delta})$ fits above the corresponding diagonal.

We illustrate this construction on this picture:



2.3 Maps G_m and G_n

Let us return to the formula (1). Given a Γ -module Δ , we can consider its m -generators a_1, \dots, a_m and compute

$$g_m(a_i) = \#([a_i, a_i + n) \setminus \Delta).$$

Theorem 2.3. ([12]) *The numbers $g_m(a_i)$ are decreasing. The Young diagram with columns $g_m(a_i)$ can be embedded in an $m \times n$ rectangle below the diagonal.*

This result allows us to consider the map G_m from the set of diagrams below the diagonal to itself, sending $D(\Delta)$ to the diagram with columns $g_m(a_i)$. We will also use the notation $G_m(\Delta) = G_m(D(\Delta))$ for a Γ -semimodule Δ . A priori, there are two different maps G_m and G_n depending on whether we choose m -generators or n -generators.

Theorem 2.4. *One has*

$$G_n(\Delta) = \left(G_m(\hat{\Delta})\right)^T.$$

In other words, the following diagram commutes:

$$\begin{array}{ccc} \Delta & \xrightarrow{G_n} & G_n(\Delta) \\ \updownarrow & \circlearrowleft & T\updownarrow \\ \hat{\Delta} & \xrightarrow{G_m} & G_m(\hat{\Delta}) \end{array}$$

Proof. Let us recall the notion of an m -cogenerator for the Γ -semimodule Δ introduced in [12]: we call b an m -cogenerator for m if $b \notin \Delta$, but $b + m \in \Delta$. Remark that by Lemma 2.2 the m -cogenerators of Δ are in 1-to-1 correspondence with the m -generators of $\hat{\Delta}$. More precisely, if b is an m -generator of $\hat{\Delta}$, then $(2\delta - 1 - b)$ is an m -cogenerator for Δ .

Let a_1, \dots, a_n be the n -generators of Δ and let b_1, \dots, b_m be the m -generators of $\hat{\Delta}$. One can check that $g_n(a_i)$ equals to the number of m -cogenerators of Δ greater than a_i :

$$g_n(a_i) = \#([a_i, a_i + m) \setminus \Delta) = \#\{j | 2\delta - 1 - b_j > a_i\} = \#\{j | a_i + b_j < 2\delta - 1\}.$$

On the other hand, analogously

$$g_m(b_j) = \#\{i | a_i + b_j < 2\delta - 1\}.$$

It is clear that the corresponding Young diagrams are transposed to each other. \square

Corollary 2.1. *If G_n is bijective then G_m is bijective.*

Corollary 2.2. *The dimensions of cells in \overline{JC}_x labelled by Δ and $\hat{\Delta}$ are the same:*

$$\dim C_\Delta = |G_n(\Delta)| = |G_m(\hat{\Delta})| = \dim C_{\hat{\Delta}}.$$

In $(n, n+1)$ case we get a priori two different maps G_{n+1} and G_n from the semigroup picture, but in combinatorial setup of [16] there is only one map known. We resolve this apparent contradiction with the following

Theorem 2.5. *If $m = n + 1$ then $G_{n+1} = G_n$.*

Proof. Let Δ be a Γ -semimodule and $x \in \Delta$. Let $a_+(x)$ be the minimal n -generator of Δ greater than or equal to x . Similarly, let $a_-(x)$ be the maximal $(n+1)$ -generator of Δ less than or equal to x .

If x is not an n -generator then $x - n \in \Delta$, so $(x - n) + (n+1) = x + 1 \in \Delta$. If $x + 1$ is not an n -generator then $x + 2 \in \Delta$ etc. By continuing this process we conclude that $[x, a_+(x)] \subset \Delta$. Since $a_+(x) - n \notin \Delta$, either $a_+(x) + 1 \notin \Delta$ or $a_+(x) + 1$ is a $(n+1)$ -generator of Δ .

Analogously, $[a_-(x), x] \subset \Delta$ and either $a_-(x) - 1 \notin \Delta$ or $a_-(x) - 1$ is an n -generator of Δ . Therefore n - and $n+1$ -generators of Δ are split into pairs (a_-, a_+) such that $[a_-, a_+] \subset \Delta$, and this is a 1-to-1 correspondence except for the largest $(n+1)$ -generator. Since $[a_-, a_+] \subset \Delta$, we have $g(a_-) = g(a_+)$. \square

Corollary 2.3. *If $m = n + 1$, then*

$$G_n(\hat{\Delta}) = (G_n(\Delta))^T.$$

Proof. By Theorems 2.4 and 2.5

$$G_n(\hat{\Delta}) = (G_{n+1}(\Delta))^T = (G_n(\Delta))^T.$$

\square

Example 2.2. *Let us present an example where G_n and G_m are essentially different. Let $(m, n) = (3, 7)$ and $\Delta = \{0, 1, 3, 4, 6, 7, 8, 9, 10, 11, 12, \dots\}$. Then the 3-generators are equal to 0, 1, 8, and*

$$g_3(0) = 2, g_3(1) = 2, g_3(8) = 0.$$

The 7-generators are 0, 1, 3, 4, 6, 9, 12, and

$$g_7(0) = g_7(1) = g_7(3) = g_7(4) = 1, g_7(6) = g_7(9) = g_7(12) = 0.$$

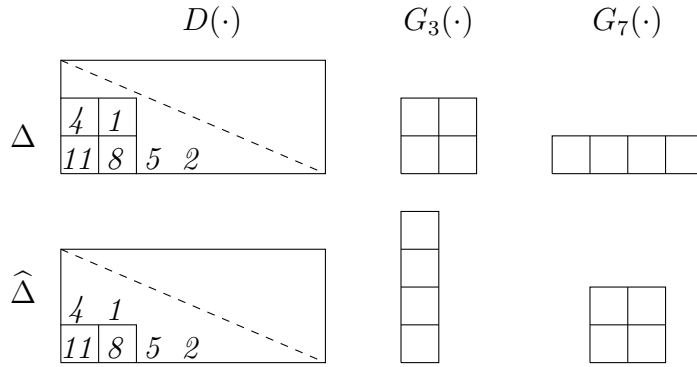
The dual semimodule is $\widehat{\Delta} = \{0, 3, 6, 7, 8, \dots\}$. Its 3-generators are 0, 7, 8, and

$$g_3(0) = 4, g_3(7) = g_3(8) = 0.$$

The 7-generators are 0, 3, 6, 8, 9, 11, 12, and

$$g_7(0) = g_7(3) = 2, g_7(6) = g_7(8) = g_7(9) = g_7(11) = g_7(12) = 0.$$

We can illustrate this on the following picture:



2.4 (m, n) -cores

Definition 2.1. A partition is called a p -core if neither of hook lengths of its boxes is equal to p .

The p -core partitions play an important role in the study of representations of symmetric groups over finite fields (see [3],[8],[29] and references therein). J. Anderson observed that the number of partitions that are simultaneously m - and n -cores is finite:

Theorem 2.6. ([3]) *The number of partitions that are simultaneously m - and n -cores equals to*

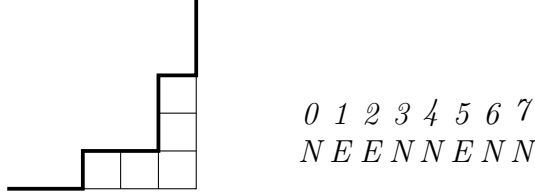
$$\frac{1}{m+n} \binom{m+n}{n}.$$

The proof in [3] uses explicit bijection between (m, n) -cores and lattice paths in $m \times n$ rectangle below the diagonal. Following the analogy between such paths and $\Gamma_{m,n}$ -semimodules, we would like to present Anderson's bijection in slightly different form.

Proof. Given a set $0 \in \Delta \subset \mathbb{Z}_{\geq 0}$, we construct the partition $P(\Delta)$ by the following rule. We start from $0 \in \Delta$ and read all consecutive integers. If $x \in \Delta$, we move north by 1, if $x \notin \Delta$, we move east by 1. The resulting lattice path bounds from above the Young diagram of the partition $P(\Delta)$.

$P(\Delta)$ has a hook of length m if and only if there are integers $x \in \Delta$ and $y \notin \Delta$ such that $y = x + m$. Therefore $P(\Delta)$ is a simultaneous (m, n) -core if and only if Δ is a $\Gamma_{m,n}$ -semimodule. \square

Example 2.3. The $(3, 4)$ -core corresponding to $\Delta = \{0, 3, 4, 6, \dots\}$ has the form:



Lemma 2.3. The (m, n) -core corresponding to $\hat{\Delta}$ is conjugate to the (m, n) -core for Δ .

Proof. Follows from Theorem 2.2 and the proof of Theorem 2.6: we essentially replace Δ by $\mathbb{Z} \setminus \Delta$, so the N steps are replaced by E steps and vice-versa. \square

Theorem 2.7. The number of self-dual Γ -semimodules equals to

$$\binom{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor}{\lfloor \frac{m}{2} \rfloor}.$$

Proof. By Lemma 2.3 the Γ -semimodules such that $\Delta = \hat{\Delta}$ correspond to the self-conjugate (m, n) -cores. The number of such cores was computed by B. Ford, H. Mai and L. Sze in [8], and it is equal to $\binom{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor}{\lfloor \frac{m}{2} \rfloor}$. \square

Remark 2.3. ([4]) The number of self-dual modules equals to

$$\binom{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor}{\lfloor \frac{m}{2} \rfloor} = \frac{[(m+n-1)!]_q}{[m!]_q [n!]_q} [q = -1].$$

Let us give a geometric interpretation of the construction from Theorem 2.6. As it was discussed in the Introduction, the vector space $V = \mathbb{C}[[t]]/t^{2\delta}\mathbb{C}[[t]]$ comes with a natural filtration

$$V = V_{2\delta} \supset V_{2\delta-1} \supset \dots \supset V_0 = 0, \quad V_i := t^{2\delta-i}V.$$

Therefore $Gr(\delta, V)$ has a natural Schubert cell decomposition, which can be described as follows.

Let P be a Young diagram contained in a $\delta \times \delta$ square. Let $p_1 \leq \dots \leq p_\delta$ be its rows. The Schubert cell $C_P \subset Gr(\delta, V)$ consists of subspaces $W \subset V$ such that

$$\dim(W \cap V_i) = \#\{j : p_j + j \leq i\}.$$

Equivalently, one can assign the diagram $P(W)$ to a subspace $W \subset V$ in the following way. Define the subset $\Delta(W) \subset \mathbb{Z}_{\geq 0}$ as follows:

$$\Delta(W) = \{d \in \mathbb{Z} : \exists p \in W, p \in V_{2\delta-d} \setminus V_{2\delta-d-1}\} \cup [t^{2\delta}, \infty).$$

Now let us construct the diagram $P(W)$ from the subset $\Delta(W)$ as in the Theorem 2.6. One can check that W belongs to the Schubert cell $C_{P(W)} \subset Gr(\delta, V)$.

Note that according to the construction, $P(W)$ is a simultaneous m, n -core iff $\Delta(W)$ is a $\Gamma_{m,n}$ -semimodule. It follows immediately that if $W \in \overline{JC}_x$ then $\Delta(W)$ is a $\Gamma_{m,n}$ -semimodule. On the other hand, J. Piontkowski showed in [27] that for a fixed $\Gamma_{m,n}$ -semimodule Δ there always exist $W \in \overline{JC}_x$ such that $\Delta(W) = \Delta$. More precisely, modules $W \in \overline{JC}_x$ with a fixed semimodule $\Delta(W)$ form a cell in the Piontkowski's cell decomposition of \overline{JC}_x .

Therefore, one gets the following

Theorem 2.8. *Let P be a Young diagram contained in a $\delta \times \delta$ square. Let $C_P \subset Gr(\delta, V)$ be the corresponding Schubert cell. Then the intersection $C_P \cap \overline{JC}_x$ is non-empty iff P is a simultaneous m, n -core, in which case it is the corresponding Piontkowski's affine cell.*

3 Bounce statistics and Poincaré polynomials

3.1 The case $m = kn + 1$.

Let Δ be a $\Gamma_{m,n}$ -semimodule. Let us recall the following definitions:

Definition 3.1. A number $a \in \Delta$ is called an m -generator if $a - m \notin \Delta$. A number $a \notin \Delta$ is called an m -cogenerator if $a + m \in \Delta$.

Let $\{0 = a_0 < \dots < a_{m-1}\} = \Delta \setminus (\Delta + m)$ be the m -generators of Δ .

Lemma 3.1. *Suppose that $x \in \Delta$ is not an m -generator. Then $x + n$ is not an m -generator as well.*

Recall that $g_m(a) = \#([a, a+n) \setminus \Delta)$.

Lemma 3.2. *Fix an integer $x \in \mathbb{Z}$. We have*

1. *The number of m -generators on $[x, x+n)$ equals to*

$$g_m(x-m) - g_m(x).$$

2. *The number of m -cogenerators on $[x, x+n)$ equals to*

$$g_m(x) - g_m(x+m).$$

Lemma 3.3. *Fix an integer $x \in \mathbb{Z}$. Number of m -generators on the interval $[x, x+n)$ is equal to the number of n -cogenerators on the interval $[x-m, x)$.*

Proof. Indeed, let A_x be the number of m -generators on an interval $[x, x+n)$, and let B_x be the number of n -cogenerators on the interval $[x-m, x)$. Then one has

$$\begin{aligned} A_x &= g_m(x-m) - g_m(x) = \#([x-m, x-m+n) \setminus \Delta) - \#([x, x+n) \setminus \Delta) = \\ &= \#([x-m, x) \setminus \Delta) - \#([x-m+n, x+n) \setminus \Delta) = g_n(x-m) - g_n(x-m+n) = B_x. \end{aligned}$$

□

Let $m = kn + 1$ from now till the end of the Section 3.1. Our goal is to reconstruct the semimodule Δ from the numbers $g_m(a_0), \dots, g_m(a_{kn})$, where a_0, \dots, a_{kn} are the m -generators of Δ .

Lemma 3.4. *Suppose that $x \in \Delta$ and $x - \alpha n - 1 \notin \Delta$ for some $\alpha \in \{0, \dots, k\}$. Then x is a $(kn+1)$ -generator.*

Proof. Indeed, if x is not a $(kn+1)$ -generator, then $x - kn - 1 \in \Delta$. But then $(x - kn - 1) + (k - \alpha)n = x - \alpha n - 1 \in \Delta$. Contradiction. □

Definition 3.2. For $x \in \Delta$ we define $a_-(x)$ to be the maximal $(kn+1)$ -generator less than or equal to x .

Corollary 3.1. *For any $x \in \Delta$ one has $[a_-(x), x] \subset \Delta$.*

Proof. Follows from Lemma 3.4 with $\alpha = 0$. □

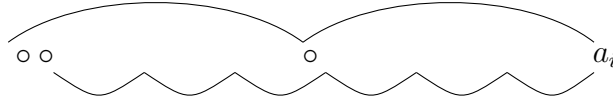
Lemma 3.5. *Consider a $(kn + 1)$ -generator a_i . Then for any $l > 0$, the interval $J_l := [a_i - l(kn + 1), a_i - lkn - 1]$ has empty intersection with Δ .*

Proof. The proof is by induction in l . The case $l = 1$ is clear. Then observe that

$$J_{l+1} = \{a_i - (l + 1)(kn + 1)\} \cup (J_l - kn).$$

□

We illustrate Lemma 3.5 with the following picture:



On this picture a_i is a $(kn + 1)$ -generator, \circ indicates an integer not in Δ , short arcs are of length n , and long arcs are of length $kn + 1$.

Definition 3.3. We introduce the following notations:

$$N_{ij} = \lceil \frac{a_j - a_i}{n} \rceil,$$

and

$$K_{ij} = \lceil \frac{a_j - a_i}{kn + 1} \rceil.$$

Corollary 3.2. *One has the following formula:*

$$K_{ij} = \lceil \frac{N_{ij}}{k} \rceil.$$

Proof. Note that

$$\lceil \frac{N_{ij}}{k} \rceil = \lceil \frac{\lceil \frac{a_j - a_i}{n} \rceil}{k} \rceil = \lceil \frac{a_j - a_i}{kn} \rceil.$$

Therefore, $K_{ij} \neq \lceil \frac{N_{ij}}{k} \rceil$ iff there exist l , such that $a_j - a_i \leq l(kn + 1)$ and $a_j - a_i > lkn$, or equivalently $a_i \in [a_j - l(kn + 1), a_j - lkn)$. So, by Lemma 3.4, $a_i \notin \Delta$. Contradiction. □

There are two steps in the reconstruction of Δ . First we reconstruct the bounce tree T_Δ , and then we use it to recover the $(kn + 1)$ -generators a_0, \dots, a_{kn} .

Definition 3.4. The oriented graph T_Δ is defined as follows. The vertices are the $(kn + 1)$ -generators of $\Delta : V_\Delta = \{a_0, \dots, a_{kn}\}$. Let $a_i \in V_\Delta$ be a $(kn + 1)$ -generator. We draw an edge $a_i \rightarrow a_j$, if $i \neq kn$ and

$$a_j = a_-(a_i + n).$$

Lemma 3.6. *The graph T_Δ satisfies the following properties:*

1. *If $a_i \rightarrow a_j$ is an edge, then $i < j$.*
2. *The graph T_Δ is a tree with the root a_{kn} , and all edges are oriented towards a_{kn} .*
3. *The leaves of T_Δ are exactly the $(kn + 1, n)$ -generators.*

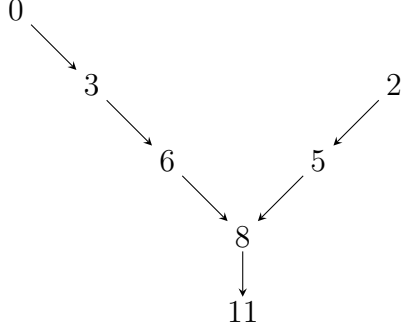
Proof. 1. By construction, $j \geq i$. If $i = j$, then there are no $(kn + 1)$ -generators on the interval $[a_i + 1, a_i + n]$, hence by Lemma 3.4 $[a_i, a_i + n] \subset \Delta$ and therefore $[a_i, +\infty) \subset \Delta$. By Lemma 3.1 there are no $(kn + 1)$ -generators greater than a_i . Therefore $i = kn$.

2. Follows from the fact that there is only one edge from each vertex and the previous part.
3. If $a_i - n \in \Delta$ then by Lemma 3.1 $a_i - n$ is a $(kn + 1)$ -generator. But then the tree T_Δ contains the edge $(a_i - n) \rightarrow a_i$ and a_i is not a leaf. Therefore, every leaf is a $(kn + 1, n)$ -generator.

Conversely, suppose that a_i is a $(kn + 1, n)$ -generator, but not a leaf of T_Δ . Then there exists a $(kn + 1)$ -generator a_j such that $a_j \rightarrow a_i$ is an edge of T_Δ , so $a_j + n > a_i$ and the interval $[a_i + 1, a_j + n]$ does not contain any $(kn + 1)$ -generators. It then follows from Lemma 3.4 that the whole interval $[a_i + 1, a_j + n]$ is contained in Δ . Therefore by Lemma 3.4 $a_i + 1 - (n + 1) = a_i - n \in \Delta$. Contradiction.

□

Example 3.1. *Consider a $\Gamma_{3,7}$ -semimodule $\Delta = \{0, 2, 3, 5, 6, 7, 8, 9, 10, 11, \dots\}$. Its 7-generators are 0, 2, 3, 5, 6, 8, 11, and the tree T_Δ has the form:*



We will also need the following observation about paths in T_Δ :

Lemma 3.7. *Suppose $a_{i_0} \rightarrow a_{i_1} \rightarrow \cdots \rightarrow a_{i_l}$ is a path in T_Δ . Then the interval $[a_{i_l} + 1, a_{i_0} + ln]$ is a subset of Δ and it does not contain $(kn + 1)$ -generators.*

Proof. The proof is by induction in l . The $l = 1$ case is by definition and Lemma 3.4. Suppose that we proved the Lemma for $l - 1$. Then one has

$$[a_{i_l} + 1, a_{i_0} + ln] = [a_{i_l} + 1, a_{i_{l-1}} + n] \cup ([a_{i_{l-1}} + 1, a_{i_0} + (l - 1)n] + n)$$

The interval $[a_{i_l} + 1, a_{i_{l-1}} + n]$ does not contain $(kn + 1)$ -generators by the definition of the edge $a_{i_{l-1}} \rightarrow a_{i_l}$. In turn, the interval $([a_{i_{l-1}} + 1, a_{i_0} + (l - 1)n] + n)$ does not contain $(kn + 1)$ -generators by the induction assumption and Lemma 3.1. Finally, the inclusion $[a_{i_l} + 1, a_{i_0} + ln] \subset \Delta$ follows from Lemma 3.4. \square

Corollary 3.3. *In the above setting, $i_l - i_0$ is exactly the number of $(kn + 1)$ -generators on the interval $[a_{i_0} + 1, a_{i_0} + ln]$, and $N_{i_0 i_l} = l$.*

Theorem 3.1. *One can reconstruct T_Δ from numbers $g_m(a_0), \dots, g_m(a_{kn})$.*

Proof. We will reconstruct T_Δ in the following order. First, we reconstruct the path from a_0 to a_{kn} . Then, we take the smallest $(kn + 1)$ -generator a_l , which is not covered yet, and reconstruct the path

$$a_l \rightarrow a_{l+b_0} \rightarrow a_{l+b_0+b_1} \rightarrow \cdots \rightarrow a_{kn}.$$

By construction, a_l is a leaf, or, equivalently, a $(kn + 1, n)$ -generator. We repeat this procedure until we run out of generators. By Corollary 3.3, b_i is the number of $(kn + 1)$ -generators on the interval $I_i := (a_l + in, a_l + (i + 1)n]$.

In the same time, by Lemma 3.3, b_i is equal to the number of n -cogenerators on the interval $J_i := (a_l + in - (kn + 1), a_l + in]$.

Note that

$$J_i = \{a_l + in - kn\} \sqcup I_{i-k} \sqcup \cdots \sqcup I_{i-1} \quad (3)$$

(here I_i for $i < 0$ is defined by the same formula).

We know that $a_l + in - kn = a_l + n(i - k)$ is an n -cogenerator if and only if $i = k - 1$. By (3) to determine b_i it is sufficient to find the numbers c_j of n -cogenerators on the intervals I_j for $j < i$:

$$b_i = \begin{cases} \sum_{j=i-k}^{j=i-1} c_j, i \neq k-1 \\ 1 + \sum_{j=i-k}^{j=i-1} c_j, i = k-1. \end{cases} \quad (4)$$

On the other hand, for $i \geq 0$ we can express c_i through b_j with $j \leq i$ using the equation

$$c_i = g(a_{l+b_0+b_1+\cdots+b_{i-1}}) - g(a_{l+b_0+b_1+\cdots+b_i}), \quad c_0 = g(a_l) - g(a_{l+b_0}), \quad (5)$$

which follows from Lemmas 3.2 and 3.7.

By (4) and (5) it suffices to find c_i for $i < 0$.

Recall the numbers $N_{ij} = \lceil \frac{a_j - a_i}{n} \rceil$ and $K_{ij} = \lceil \frac{a_j - a_i}{kn+1} \rceil$. Since we know the edges of the tree T_Δ going from $(kn+1)$ -generators less than a_l , by Corollary 3.3, we know N_{ij} for any $i, j \leq l$. Then, by Corollary 3.2 we also know numbers K_{ij} for any $i, j \leq l$. Note that

$$f(a_i) := \#(\Delta \cap (-\infty, a_i)) = \sum_{j < i} K_{ij}.$$

Indeed, fix a reminder $0 \leq r < kn + 1$, such that the corresponding $(kn+1)$ -generator $a_j \equiv r \pmod{kn+1}$ is less than a_i . Then there are exactly K_{ij} integers in $\Delta \cap (-\infty, a_i)$ with reminder r modulo $kn+1$.

Let us define (for $i \geq 0$) $\alpha_i := \min\{j | N_{jl} = i\}$ (i. e. a_{α_i} is the smallest $(kn+1)$ -generator greater than or equal to $a_l - in$). Then for $i > 0$

$$g_m(a_l - in) = n - (f(a_{\alpha_{i-1}}) - f(a_{\alpha_i})).$$

For $i = 0$ we know $g_m(a_l)$, thus we can compute

$$c_{-i} = g_m(a_l - in) - g_m(a_l - in + n) \text{ for all } i > 0.$$

□

Theorem 3.2. *The tree T_Δ completely determines the semimodule Δ .*

Proof. First of all, by looking at the path from a_0 to a_{kn} , we immediately derive how many $(kn+1)$ -generators are on each interval $(Nn, (N+1)n]$, $N \geq 0$. Namely, $a_i \in [(K_{0i}-1)(nk+1), K_{0i}(nk+1))$ and, as it was noticed above, we can compute numbers K_{ij} using the tree T_Δ and Lemma 3.5.

Let r_0, \dots, r_{kn} be the reminders modulo $kn+1$ of a_0, \dots, a_{kn} respectively. We can recover the order of r_0, \dots, r_{kn} . Indeed, if $i < j$

$$r_i < r_j \Leftrightarrow K_{ij} > K_{0j} - K_{0i}.$$

Since the reminders r_0, \dots, r_{kn} run through all numbers $0, 1, \dots, kn$ once, knowing the order of r_0, \dots, r_{kn} is equivalent to knowing the reminders themselves. Combining this with the fact that we know to which interval $[p(kn+1), (p+1)(kn+1))$ each $(kn+1)$ -generator belongs, we recover all the $(kn+1)$ -generators a_0, \dots, a_{kn} . □

3.2 The case $m = kn - 1$.

Let now $m = kn - 1$. In this case the semimodule Δ can be reconstructed from numbers $g_m(a_0), \dots, g_m(a_{m-1})$ in a way similar to the case $m = kn + 1$. However, some adjustments are required. We will omit some of the proofs, in the cases when they are identical to the $m = kn + 1$ case.

Lemma 3.8. *Suppose that $x \in \Delta$ and $x - \alpha n + 1 \notin \Delta$ for some $\alpha \in \{0, \dots, k\}$. Then x is a $(kn - 1)$ -generator.*

Proof. Indeed, if x is not a $(kn - 1)$ -generator, then $x - kn + 1 \in \Delta$. But then $(x - kn + 1) + (k - \alpha)n = x - \alpha n + 1 \in \Delta$. Contradiction. □

Definition 3.5. For $x \in \Delta$ we define $a_+(x)$ to be the minimal $(kn - 1)$ -generator greater than or equal to x .

Corollary 3.4. *For any $x \in \Delta$ one has $[x, a_+(x)] \subset \Delta$.*

Proof. Follows from Lemma 3.8 with $\alpha = 0$. □

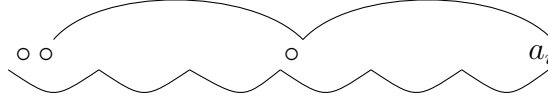
Lemma 3.9. *Consider a $(kn - 1)$ -generator a_i . Then for any $l > 0$, the interval $J_l := [a_i - lkn + 1, a_i - l(kn - 1)]$ has empty intersection with Δ .*

Proof. The proof is by induction in l . The case $l = 1$ is clear. Then observe that

$$J_{l+1} = \{a_i - (l+1)(kn-1)\} \cup (J_l - kn).$$

□

We illustrate Lemma 3.9 with the following picture:



On this picture a_i is a $(kn-1)$ -generator, \circ indicates an integer not in Δ , short arcs are of length n , and long arcs are of length $kn-1$.

Definition 3.6. We introduce the following notations:

$$N_{ij} = \lfloor \frac{a_j - a_i}{n} \rfloor + 1,$$

and

$$K_{ij} = \lfloor \frac{a_j - a_i}{kn-1} \rfloor + 1.$$

Corollary 3.5. We have the following formula:

$$K_{ij} = \lceil \frac{N_{ij}}{k} \rceil.$$

Proof. Note that

$$\lceil \frac{N_{ij}}{k} \rceil = \lceil \frac{\lfloor \frac{a_j - a_i}{n} \rfloor + 1}{k} \rceil = \lfloor \frac{a_j - a_i}{kn} \rfloor + 1.$$

Therefore, $K_{ij} \neq \lceil \frac{N_{ij}}{k} \rceil$ iff there exist l , such that $a_j - a_i \geq l(kn-1)$ and $a_j - a_i < lkn$, or equivalently $a_i \in (a_j - lkn, a_j - l(kn-1)]$. So, by Lemma 3.8, $a_i \notin \Delta$. Contradiction. □

The definition of the bounce tree T_Δ should be slightly adjusted compare to the definition in the $m = kn + 1$ case:

Definition 3.7. The oriented graph T_Δ is defined as follows. The vertices are the $(kn-1)$ -generators of Δ plus one extra vertex $a_\infty : V_\Delta =$

$\{a_0, \dots, a_{kn-2}, a_\infty\}$. Let $a_i \in V_\Delta$ be a $(kn-1)$ -generator. We draw an edge $a_i \rightarrow a_j$, if

$$a_j = a_+(a_i + n).$$

In addition, we draw edges $a_i \rightarrow a_\infty$ for every a_i , such that there is no $(kn-1)$ -generators greater than or equal to $a_i + n$.

Lemma 3.10. *The graph T_Δ satisfies the following properties:*

1. *If $a_i \rightarrow a_j$ is an edge, then $i < j$.*
2. *The graph T_Δ is a tree with the root a_∞ , and all edges are oriented towards a_∞ .*
3. *The leaves of T_Δ are exactly the $(kn-1, n)$ -generators.*

Proof. 1. By construction.

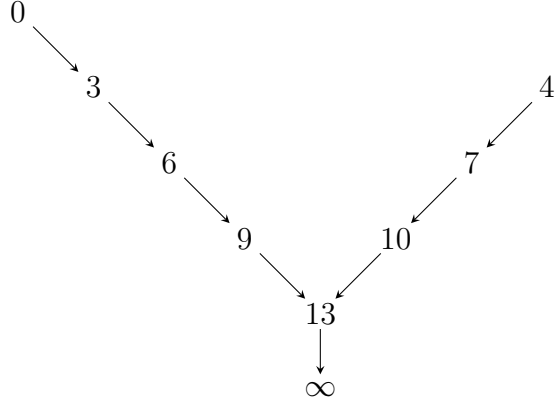
2. Also immediate from the construction.

3. If $a_i - n \in \Delta$ then by Lemma 3.1 $a_i - n$ is a $(kn-1)$ -generator. But then the tree T_Δ contains the edge $(a_i - n) \rightarrow a_i$ and a_i is not a leaf. Therefore, every leaf is a $(kn-1, n)$ -generator.

Conversely, suppose that a_i is a $(kn-1, n)$ -generator, but not a leaf of T_Δ . Then there exists a $(kn-1)$ -generator a_j such that $a_j \rightarrow a_i$ is an edge of T_Δ , so $a_j + n < a_i$ and the interval $[a_j + n, a_i - 1]$ does not contain any $(kn-1)$ -generators. It then follows from Lemma 3.8 that the whole interval $[a_j + n, a_i - 1]$ is contained in Δ . Therefore by Lemma 3.8 $a_i - 1 - (n-1) = a_i - n \in \Delta$. Contradiction.

□

Example 3.2. *Consider a $\Gamma_{3,8}$ -semimodule $\Delta = \{0, 3, 4, 6, 7, 8, 9, \dots\}$, then its 8-generators are 0, 3, 4, 6, 7, 9, 10, 13, and the tree T_Δ has the form:*



We will also need the following observation about paths in T_Δ :

Lemma 3.11. *Suppose $a_{i_0} \rightarrow a_{i_1} \rightarrow \cdots \rightarrow a_{i_l}$ is a path in T_Δ . Then the interval $[a_{i_0} + ln, a_{i_l} - 1]$ is a subset of Δ and it does not contain $(kn - 1)$ -generators.*

Proof. The proof is by induction in l . The $l = 1$ case is by definition and Lemma 3.8. Suppose that we proved the Lemma for $l - 1$. Then one has

$$[a_{i_0} + ln, a_{i_l} - 1] = ([a_{i_0} + (l - 1)n, a_{i_l} - 1] + n) \cup [a_{i_{l-1}} + n, a_{i_l} - 1]$$

The interval $[a_{i_{l-1}} + n, a_{i_l} - 1]$ does not contain $(kn - 1)$ -generators by the definition of the edge $a_{i_{l-1}} \rightarrow a_{i_l}$. In turn, the interval $([a_{i_0} + (l - 1)n, a_{i_l} - 1] + n)$ does not contain $(kn - 1)$ -generators by the induction assumption and Lemma 3.1. Finally, the inclusion $[a_{i_0} + ln, a_{i_l} - 1] \subset \Delta$ follows from Lemma 3.8. \square

Corollary 3.6. *In the above setting, $i_l - i_0$ is exactly the number of $(kn - 1)$ -generators on the interval $[a_{i_0}, a_{i_0} + ln - 1]$, and $N_{i_0 i_l} = l + 1$.*

Theorem 3.3. *One can reconstruct T_Δ from numbers $g_m(a_0), \dots, g_m(a_{kn-2})$.*

Proof. We will reconstruct T_Δ in the following order. First, we reconstruct the path from a_0 to a_∞ . Then, we take the smallest $(kn - 1)$ -generator a_l , which is not covered yet, and reconstruct the path

$$a_l \rightarrow a_{l+b_0} \rightarrow a_{l+b_0+b_1} \rightarrow \cdots \rightarrow a_\infty.$$

By construction, a_l is a leaf, or, equivalently, a $(kn - 1, n)$ -generator. We repeat this procedure until we run out of generators. By Corollary 3.6, b_i is

the number of $(kn-1)$ -generators on the interval $I_i := [a_l + in, a_l + (i+1)n)$. In the same time, by Lemma 3.3, b_i is equal to the number of n -cogenerators on the interval $J_i := [a_l + in - (kn+1), a_l + in)$.

Note that

$$J_i = (I_{i-k} \sqcup \cdots \sqcup I_{i-1}) \setminus (a_l + (i-k)n) \quad (6)$$

(here I_i for $i < 0$ is defined by the same formula).

We know that $a_l + in - kn = a_l + n(i-k)$ is an n -cogenerator if and only if $i = k-1$. By (6) to determine b_i it is sufficient to find the numbers c_j of n -cogenerators on the intervals I_j for $j < i$:

$$b_i = \begin{cases} \sum_{j=i-k}^{j=i-1} c_j, & i \neq k-1 \\ -1 + \sum_{j=i-k}^{j=i-1} c_j, & i = k-1. \end{cases} \quad (7)$$

On the other hand, for $i \geq 0$ we can express c_i through b_j with $j \leq i$ using the equation

$$c_i = g(a_{l+b_0+b_1+\cdots+b_{i-1}}) - g(a_{l+b_0+b_1+\cdots+b_i}), \quad c_0 = g(a_l) - g(a_{l+b_0}), \quad (8)$$

which follows from Lemmas 3.2 and 3.11.

By (7) and (8) it suffices to find c_i for $i < 0$.

Recall the numbers $N_{ij} = \lfloor \frac{a_j - a_i}{n} \rfloor + 1$ and $K_{ij} = \lfloor \frac{a_j - a_i}{kn+1} \rfloor + 1$. Since we know the edges of the tree T_Δ going from $(kn-1)$ -generators less than a_l , by Corollary 3.6, we know N_{ij} for any $i, j \leq l$. Then, by Corollary 3.5 we also know numbers K_{ij} for any $i, j \leq l$. Note that

$$f(a_i) := \sharp(\Delta \cap (-\infty, a_i]) = \sum_{j \leq i} K_{ij}.$$

Indeed, fix a reminder $0 \leq r < kn-1$, such that the corresponding $(kn-1)$ -generator $a_j \equiv r \pmod{kn-1}$ is less than or equal to a_i . Then there are exactly K_{ij} integers in $\Delta \cap (-\infty, a_i]$ with reminder r modulo $kn-1$.

Let us define (for $i \geq 0$) $\alpha_i := \max\{j | N_{jl} = i\}$ (i. e. a_{α_i} is the biggest $(kn-1)$ -generator less than or equal to $a_l - in$). Then for $i > 0$

$$\sharp([a_l - in, a_l - (i-1)n) \setminus \Delta) = n - (f(a_{\alpha_{i-1}}) - f(a_{\alpha_i})).$$

For $i = 0$ we know $\sharp([a_l, a_l + n] \setminus \Delta) = \sharp((a_l, a_l + n] \setminus \Delta) = g_m(a_l)$, thus we can compute

$$c_{-i} = \sharp([a_l - in, a_l - (i-1)n] \setminus \Delta) - \sharp([a_l - (i-1)n, a_l - (i-2)n] \setminus \Delta)$$

for all $i > 0$.

□

Theorem 3.4. *The tree T_Δ completely determines the semimodule Δ .*

Proof. The same as in the $m = kn + 1$ case.

□

3.3 Bounce path and statistic.

In the case $m = kn + 1$ the path $a_0 \rightarrow a_{b_0} \rightarrow a_{b_0+b_1} \rightarrow \cdots \rightarrow a_{b_0+\cdots+b_s}$ in the tree T_Δ should be compared with the *bounce path*, constructed by J. Haglund in the case $m = n + 1$ and generalized by N. Loehr for the case $m = kn + 1$. Basically, the numbers b_0, \dots, b_s are the horizontal steps in the bounce path. Here we recall Loehr's definition and check that it matches the path from a_0 to a_{kn} on the tree T_Δ . We also generalize the bounce path and statistic to the case $m = kn - 1$ by considering the path from a_0 to a_∞ .

Definition 3.8. ([20]) Let D be a Young diagram contained below the diagonal in an $m \times n$ -rectangle. Let $m = kn + 1$. The bounce path is defined as follows. We start from the northwest corner. We alternate southward and eastward steps with the first step going southward. We always stay outside the diagram D .

On each southward step we go south till we hit a horizontal piece of the boundary of D . Each eastward step is equal to the sum of the last k southward steps (if there were less than k southward steps yet, then it is equal to the sum of all preceding southward steps).

Let us introduce the coordinates so that the southwest corner is $(0, 0)$. Then the bounce path starts from $(0, n)$ and finishes at $(kn, 0)$. The bounce statistic is defined as follows:

Definition 3.9. [20] Let (v_0, \dots, v_a) and (h_0, \dots, h_a) be the vertical and horizontal steps of the bounce path correspondingly. Then the statistic $\text{bounce}(D)$ is defined by the formula:

$$\text{bounce}(D) := (n - v_0) + (n - v_0 - v_1) + \cdots + (n - \sum_{0 \leq i \leq a} v_i).$$

In other word, $\text{bounce}(D)$ is equal to the sum of vertical coordinates of the southwest corners of the bounce path.

Let now Δ be a semimodule over $\Gamma_{m,n}$, $m = kn + 1$. Let D be the corresponding Young diagram, and, as in the Section 2.3, $G_m(D)$ be the diagram with columns $g_m(a_0), \dots, g_m(a_{kn})$.

Let T_Δ be the bounce tree, and $a_0 \rightarrow a_{b_0} \rightarrow \dots \rightarrow a_{b_0+\dots+b_s} = a_{kn}$ be the path from a_0 to a_{kn} on it.

Theorem 3.5. *The horizontal steps (h_0, \dots, h_a) of the bounce path for $G_{kn+1}(D)$ are exactly (b_0, \dots, b_s) . In particular, $a = s$.*

Proof. Recall the formulae (4) and (5) (we plug $l = 0$):

$$b_i = \begin{cases} \sum_{j=i-k}^{j=i-1} c_j, i \neq k-1 \\ 1 + \sum_{j=i-k}^{j=i-1} c_j, i = k-1. \end{cases}$$

and

$$c_i = g(a_{b_0+b_1+\dots+b_{i-1}}) - g(a_{b_0+b_1+\dots+b_i}), \quad c_0 = g(a_0) - g(a_{b_0}).$$

where c_i is the number of n -cogenerators on the interval $(in, (i+1)n]$.

We immediately see that

$$c_i = \begin{cases} 0, i < -2 \\ 1, i = -2 \\ n - g_m(a_0) - 1, i = -1 \end{cases}$$

Indeed, $-n$ is the smallest n -cogenerator, and there are exactly $n - g_m(a_0)$ n -cogenerators less than zero.

For $i \geq 0$ the recurrence relations on the numbers c_i and b_i are almost the same as the definitions of the vertical and horizontal steps of the bounce path correspondingly. There are two differences:

1. The first vertical step equals to $v_0 = n - g_m(a_0) = c_{-1} + 1 = c_{-1} + c_{-2}$.
2. For $i = k-1$ one has $b_{k-1} = 1 + \sum_{j=-1}^{j=k-2} c_j = \sum_{j=-2}^{j=k-2} c_j$.

One immediately sees that those differences cancel each other. Therefore one gets

$$v_i = \begin{cases} c_{i-1}, i > 0 \\ c_{-1} + c_{-2}, i = 0 \end{cases}$$

and

$$h_i = b_i$$

for all $i \geq 0$. □

Theorem 3.6. *We get the following relation:*

$$\text{bounce}(G_{kn+1}(D)) = \delta - |D|.$$

Proof. Indeed, we have

$$\delta - |D| = \sharp(\mathbb{Z}_{>0} \setminus \Delta) = \sum_{i=0}^{\infty} \sharp((in, (i+1)n] \setminus \Delta).$$

By Lemma 3.7, $[a_{b_0+\dots+b_i}, ni] \subset \Delta$. We get

$$\sharp((in, (i+1)n] \setminus \Delta) = g_m(a_{b_0+\dots+b_{i-1}}) \text{ and } \sharp((0, n] \setminus \Delta) = g_m(a_0).$$

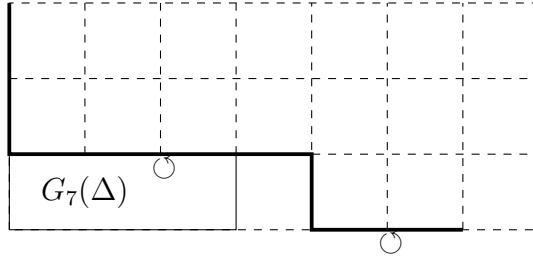
Therefore,

$$\delta - |D| = g_m(a_0) + g_m(a_{b_0}) + g_m(a_{b_0+b_1}) + \dots + g_m(a_{kn}),$$

Note that this matches the definition of the bounce statistic. □

Remark 3.1. *It follows from the proof of Theorem 3.6 that the number elements in $[in, (i+1)n] \setminus \Delta$ equals to the vertical coordinate of the i -th southwest corner of the bounce path.*

Example 3.3. *Let us return to the $(3,7)$ -module $\Delta = \{0, 2, 3, 5, 6, 7, 8, 9, \dots\}$ from Example 3.1. The 7-generators are $0, 2, 3, 5, 6, 8, 11$, so the diagram $G_7(\Delta)$ with the bounce path has the form:*



We use the above observations to generalize the bounce path and statistic to the case $m = kn - 1$.

Let Δ be a $\Gamma_{m,n}$ -semimodule, $m = kn - 1$, $D = D(\Delta)$. Consider the bounce tree T_Δ . Let $a_0 \rightarrow a_{b_0} \rightarrow \cdots \rightarrow a_{b_0+\cdots+b_s} \rightarrow a_\infty$ be the path from $a_0 = 0$ to a_∞ on the tree T_Δ . Consider the Young diagram $G_m(D)$ embedded in the $m \times n$ rectangle below the diagonal.

Definition 3.10. The bounce path for $G_m(D)$ starts at the northwest corner. It consists of alternating southward and eastward steps, starting with a southward step.

On each southward step we go south till we hit a horizontal piece of the boundary of $G_m(D)$. The i th eastward step equals to b_i , starting with $i = 0$.

The bounce statistic is defined in the same way as in the $m = kn + 1$ case:

Definition 3.11. Let (v_0, \dots, v_s) and (h_0, \dots, h_s) be the vertical and horizontal steps of the bounce path correspondingly. Then the bounce($G_m(D)$) is given by

$$\text{bounce}(G_m(D)) = (n - v_0) + (n - v_0 - v_1) + \cdots + (n - \sum_{0 \leq i \leq s} v_i).$$

In other word, bounce($G_m(D)$) is equal to the sum of vertical coordinates of the southwest corners of the bounce path.

The following formula relating the bounce of $G_m(D)$ with the area of D is proved in the same way as in the $m = kn + 1$ case:

Theorem 3.7. *We get the following relation:*

$$\text{bounce}(G_{kn-1}(D)) = \delta - |D|.$$

Finally, we expand the recurrent relations involved in the definition of the bounce path in this case to get a simple description of the bounce path in terms of the diagram $G_m(D)$. This is parallel with the proof of the Theorem 3.5 in the $m = kn + 1$ case.

Recall the formulae (7) and (8) (we plug $l = 0$):

$$b_i = \begin{cases} \sum_{j=i-k}^{j=i-1} c_j, & i \neq k-1 \\ -1 + \sum_{j=i-k}^{j=i-1} c_j, & i = k-1. \end{cases}$$

And

$$c_i = g(a_{b_0+b_1+\dots+b_{i-1}}) - g(a_{b_0+b_1+\dots+b_i}), \quad c_0 = g(a_0) - g(a_{b_0}),$$

where c_i is the number of n -cogenerators on the interval $[in, (i+1)n)$.

We immediately see that

$$c_i = \begin{cases} 0, & i < -1 \\ n - g_m(a_0), & i = -1 \end{cases}$$

Indeed, $-n$ is the smallest n -cogenerator, and there are exactly $n - g_m(a_0)$ n -cogenerators less than zero.

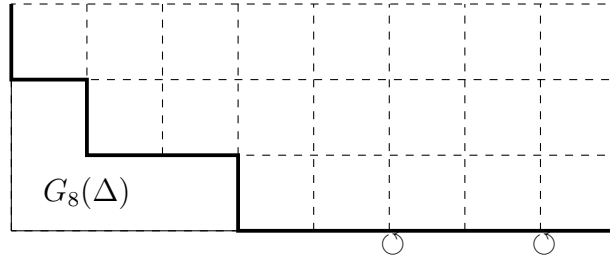
Comparing the recurrent relations one immediately gets

$$v_i = c_{i-1} \quad \text{and} \quad h_i = b_i \quad \text{for all} \quad i \geq 0.$$

Therefore, the bounce path in the $m = kn - 1$ case is constructed in the same way as in the $m = kn + 1$ case, except that the $(k-1)$ -th horizontal step is shorter by 1 (we count steps starting from 0):

$$h_i = \begin{cases} \sum_{j=i-k}^{j=i-1} v_j, & i \neq k-1 \\ -1 + \sum_{j=i-k}^{j=i-1} v_j, & i = k-1. \end{cases}$$

Example 3.4. Let us return to the $(3,8)$ -module $\Delta = \{0, 3, 4, 6, 7, 8, 9, \dots\}$ from Example 3.2. The 8-generators are 0, 3, 4, 6, 7, 9, 10, 13, so the diagram $G_8(\Delta)$ with the bounce path has the form:



4 Monomial bases and (q, t) – symmetry

4.1 Algebraic model

Following [7], we consider the quotient $A_{m,n}$ of the polynomial ring in variables $u_2, \dots, u_n, v_2, \dots, v_m$ by the ideal generated by coefficients of the z -

expansion of the equation

$$(1 + u_2 z^2 + \dots + u_n z^n)^m = (1 + v_2 z^2 + \dots + v_m z^m)^n.$$

There are several useful facts ([7]) about this ring:

1. It is a 0-dimensional complete intersection of multiplicity $\frac{(m+n-1)!}{m!n!}$.
2. The equations are homogeneous in the following grading: $q(u_i) = q(v_i) = i$.
3. The variables v_i (or u_i) can be eliminated by writing

$$(1 + u_2 z^2 + \dots + u_n z^n)^{\frac{m}{n}} = (1 + v_2 z^2 + \dots + v_m z^m).$$

Therefore $A_{m,n}$ can be realized in the following way:

$$A_{m,n} = \mathbb{C}[u_2, \dots, u_n] / (\text{Coef}_{\geq m+1} [(1 + u_2 z^2 + \dots + u_n z^n)^{\frac{m}{n}}]).$$

Although this reformulation is not (m, n) -symmetric, it is useful for the computations. In fact, one can show (e.g [11]) that the defining ideal is generated by the coefficients at $z^{m+1}, \dots, z^{m+n-1}$, and all further coefficients can be expressed through them.

The ring $A_{m,n}$ is equipped with the q -grading and the filtration by the powers of the maximal ideal \mathfrak{m} . More precisely, for $f \in A_{m,n}$ we define

$$m(f) = \max\{k | f \in \mathfrak{m}^k\}.$$

Conjecture 4.1. *There exists a monomial basis in $A_{m,n}$ such that*

1. *For every basic monomial ϕ there exists a diagram $D(\phi)$ labelling some cell in the Jacobi factor \overline{JC}_x , and this correspondence is bijective.*
2. *There is a correspondence between the pairs of statistics:*

$$q(\phi) = |D(\phi)| + h_+(D(\phi)), \quad m(\phi) = h_+(D(\phi)).$$

The following corollary was conjectured by L. Göttsche (unpublished):

Corollary 4.1. *The Poincaré polynomial of the compactified Jacobian equals to*

$$\begin{aligned} P_{\overline{JC}_x}(t) &= \sum_D t^{2 \dim C_D} = \sum_D t^{2\delta - 2h_+(D)} = t^{2\delta} \sum_{\phi} t^{-2m(\phi)} = \\ &= t^{(m-1)(n-1)} \sum_{k=0}^{\infty} t^{-2k} \dim \mathfrak{m}^k / \mathfrak{m}^{k+1}. \end{aligned}$$

In the two following subsections we prove this conjecture for $(2, m)$ and $(3, m)$ singularities.

4.2 $(2, m)$ case

One can write

$$A_{2,2k+1} = C[u_2] / (\text{Coef}_{2k+2}(1 + u_2 z^2)^{\frac{2k+1}{2}}) = C[u_2] / (u_2^{k+1}).$$

This quotient is generated by the monomials $u_2^i, i \leq k$ and

$$q(u_2^i) = 2i, m(u_2^i) = i.$$

In the combinatorial model we consider Young diagrams in $2 \times (2k+1)$ rectangle below the diagonal. Such a diagram D_i can have only one row with $i \leq k$ boxes in it, and $|D_i| = h_+(D_i) = i$.

The correspondence between diagrams and monomials identifies u_2^i and D_i .

4.3 $(3, m)$ case

The case $(3, m)$ turns out to be more subtle. In the algebraic model we have now two generators u_2 and u_3 .

Lemma 4.1. *The ring $A_{3,m}$ has a monomial basis*

$$\{u_2^a u_3^b, a + 3b \leq m - 1\}. \quad (9)$$

The gradings can be computed as

$$q(u_2^a u_3^b) = 2a + 3b, \quad m(u_2^a u_3^b) = a + b.$$

Proof. Let us consider the case $m = 3k + 1$, the case $m = 3k + 2$ is analogous. The defining equations p_{3k+2} and p_{3k+3} have degrees $3k + 2$ and $3k + 3$ respectively. One can check that p_{3k+3} has a non-zero coefficient at u_3^{k+1} and p_{3k+2} has a non-zero coefficient at $u_3^k u_2$. The syzygy between leading monomials shows that the leading monomial in

$$p_{3k+5} := u_2 p_{3k+3} - \lambda u_3 p_{3k+2} = u_2 (c_1 u_3^{k+1} + c_2 u_3^{k-1} u_2^3 + \dots) - \frac{c_1}{d_1} u_3 (d_1 u_3^k u_2 + d_2 u_3^{k-2} u_2^4 + \dots)$$

equals to $u_3^{k-1} u_2^4$. The syzygy between leading monomials in p_{3k+2} and p_{3k+5} has a form

$$p_{3k+8} := u_3 p_{3k+5} - \lambda u_2^3 p_{3k+2},$$

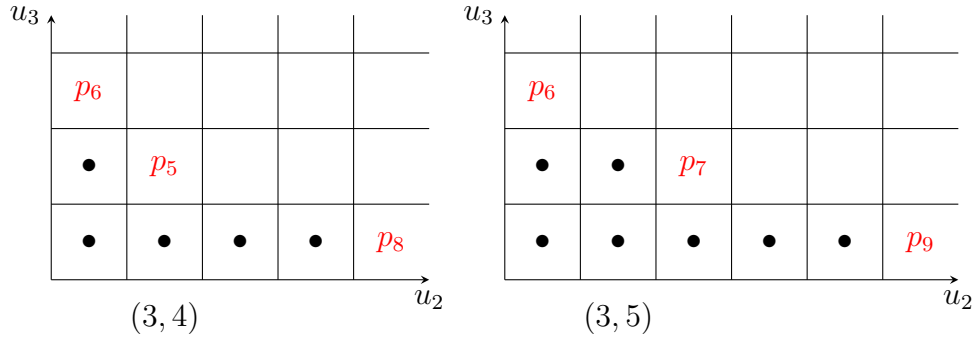
and its leading monomial is $u_3^{k-2} u_2^7$ etc.

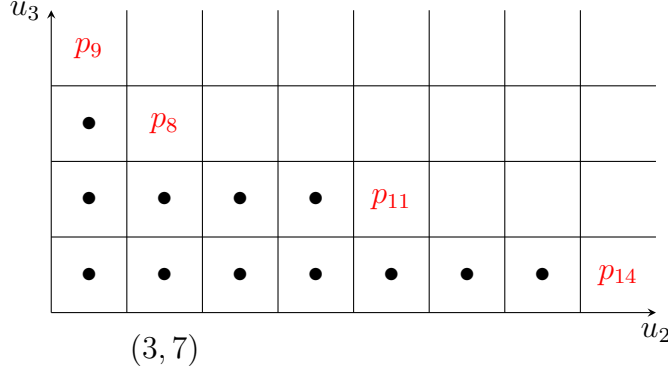
Using this process, we can eliminate the monomials

$$u_3^{k+1}, u_3^k u_2, u_3^{k-1} u_2^4, u_3^{k-2} u_2^7, \dots u_2^{3k+1}.$$

In the quotient we get the monomials (9). Since the number of these monomials equals to $\dim A_{m,n}$, there are no further relations, and these monomials form a basis in $A_{m,n}$. \square

The bases for the cases $(3, 4)$, $(3, 5)$, and $(3, 7)$ are illustrated on the following diagrams:





Theorem 4.1. *The Conjecture 4.1 holds for $n = 3$. The bigraded character $c_{3,m}(q, t)$ is symmetric in q and t . More precisely, there exist a bijection ι on the cells of \overline{JC}_x such that*

$$|\iota(D)| = \delta - h_+(\iota(D)), \quad h_+(D) = \delta - |\iota(D)|.$$

Proof. Let us prove the Conjecture 4.1 in this case.

We consider Young diagrams in $3 \times m$ rectangle below the diagonal. Such a digram $D_{\alpha,\beta}$ has two rows of length α and β , $\alpha \leq \beta$. Moreover,

$$\alpha \leq k := \lfloor \frac{m}{3} \rfloor, \quad \beta \leq \lfloor \frac{2m}{3} \rfloor.$$

The area of $D_{\alpha,\beta}$ equals to $\alpha + \beta$. To compute the statistics $h_+(D_{\alpha,\beta})$, we have to consider three cases:

1) $\beta \leq k$. In this case one can check that $h_+(D_{\alpha,\beta}) = \beta$. The corresponding monomial is equal to

$$D_{\alpha,\beta} \longleftrightarrow u_2^{\beta-\alpha} u_3^\alpha.$$

Remark that we get all monomials $u_2^a u_3^b$ with $a + b \leq k$.

2) $\beta > k, \beta - \alpha \leq k$. In this case one can check that $h_+(D_{\alpha,\beta}) = 2\beta - k$. The corresponding monomial is equal to

$$D_{\alpha,\beta} \longleftrightarrow u_2^{3\beta-2k-\alpha} u_3^{\alpha-\beta+k}.$$

Remark that we get all monomials $u_2^a u_3^b$ such that $a + b > k, 3b + a \leq m - 1$, and $a + b + k$ even.

3) $\beta - \alpha > k$. In this case one can check that $h_+(D_{\alpha,\beta}) = 2\alpha + k + 1$. The corresponding monomial is equal to

$$D_{\alpha,\beta} \longleftrightarrow u_2^{3\alpha-\beta+2k+2} u_3^{\beta-\alpha-k-1}.$$

Remark that we get all monomials $u_2^a u_3^b$ such that $a + b > k$, $3b + a \leq m - 1$, and $a + b + k$ odd.

Note that combining the three above cases we got every monomial $u_2^a u_3^b$ with $3b + a \leq m - 1$ exactly once. Therefore the first part of the Theorem is proved.

To give a bijective proof of the q, t -symmetry, one has to compose the following maps:

- 1) Given a diagram $D_{\alpha,\beta}$, construct a corresponding monomial $\phi = u_2^a u_3^b$.
- 2) Transform it to the "dual" monomial $\widehat{\phi} = u_2^{\delta-a-3b} u_3^b$.
- 3) Consider the diagram $D_{\alpha',\beta'}$ for this monomial.

By construction,

$$q(\widehat{\phi}) = 2\delta - q(\phi), \quad m(\widehat{\phi}) = m(\phi) + \delta - q(\phi),$$

Therefore

$$|D_{\alpha,\beta}| = \delta - h_+(D_{\alpha',\beta'}), \quad h_+(D_{\alpha,\beta}) = \delta - |D_{\alpha',\beta'}|.$$

□

4.4 $(4, m)$ case

The case $(4, m)$ is more complicated since both degrees of u_3^2 and $u_4 u_2$ are equal. We choose the partial order on monomials such that $u_3^2 \prec u_4 u_2$. Based on the experimental data, we propose the following:

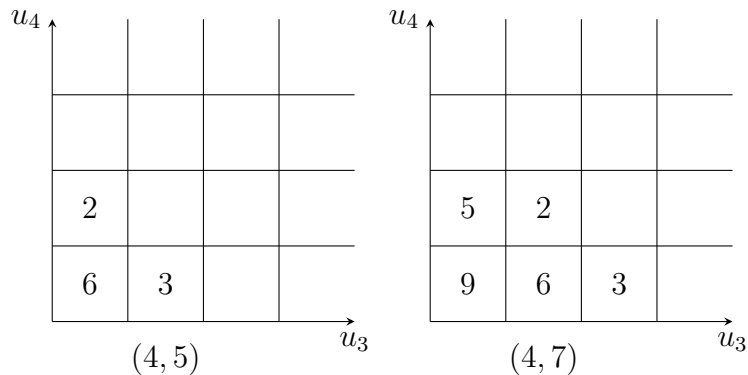
Conjecture 4.2. *The ring $A_{4,4k+1}$ has a monomial basis consisting of monomials*

$$u_2^a u_3^b u_4^c, \quad a \leq 6k - 3b - 4c, \quad b \leq k - c + 2 \lfloor \frac{k-c}{2} \rfloor. \quad (10)$$

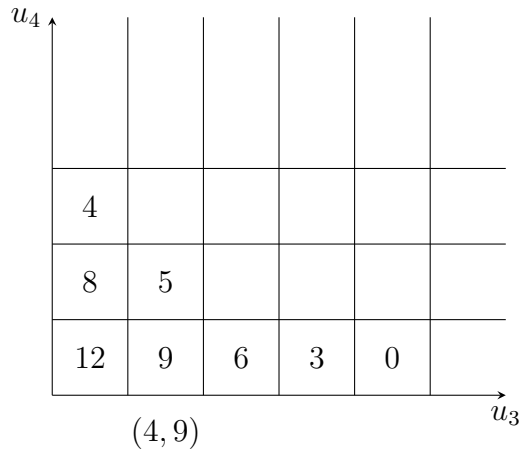
The ring $A_{4,4k+3}$ has a monomial basis consisting of monomials

$$u_2^a u_3^b u_4^c, \quad a \leq 6k - 3b - 4c, \quad b \leq k - c + 2 \lfloor \frac{k-c}{2} \rfloor + 1. \quad (11)$$

We can illustrate these monomial bases by drawing the maximal powers of u_2 on the (u_3, u_4) plane:



The corresponding dimensions are equal to $7 + 4 + 3 = 14$ and $10 + 7 + 4 + 6 + 3 = 30$.



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